Training course for the Oxbridge interviews in December

Topic 0: sample notes

Outline. This sample contains 3 Oxbridge interview problems on different topics. Problem 1.14 is given with general hints and a detailed solution. Problem 1.16 is given with general and step-by-step hints. We are happy to mark your solution to first homework Problem 1.16 for free. If Problem 1.16 seems too easy, submit Problem 4.16, which is given without hints.

The full notes on each of Topics 1–8 contain 16 training problems with general hints and solutions, plus one longer homework problem with general and step-by-step hints. Problems 8.9–8.16 in the final notes will be marked as a 1-hour mock exam. If you find any mistakes or typos, please e-mail Dr Olga Anosova, master.maths.tutor@gmail.com.

Recall that, for any integer $n \geq 2$, another integer m has the *remainder* $r \in \{0, 1, \ldots, n-1\}$ *modulo* n if $m = kn + r$ for some integer k. For instance, any even integer has the remainder 0 modulo 2, any odd integer has the remainder 1 modulo 2. So the number 0 is *even* by definition, which answers the popular question asked by many Oxbridge candidates.

Problem 1.14. Find all integer solutions (x, y) of the equation
$7566271829x^2 + 19384736483xy + 362748575y^2 = 0.$

Problem 1.16. What are the last two digits of the number 9999!! that is obtained by multiplying all odd numbers from 1 to 9999?

Problem 4.16. How many stationary points does $y(x) = x^{1/(x^x)}$ have?

General hints on Problem 1.14 (other methods are possible)
What can you say about divisibility of the integers x and y by 2?

General hints on Problem 1.16 (other methods are possible)
The problem is to compute $9999!! = 1 \cdot 3 \cdots 9999$ modulo $100 = 25 \cdot 4$.

Solution to Problem 1.14 (other methods are possible)
The trivial integer solution is $x = y = 0$. Looking for other integer solutions, notice that both x, y should be even since all the coefficients are odd. Indeed, if both x, y are odd, the sum of three odd integers can't be 0.

$$7566271829x^2 + 19384736483xy + 362748575y^2 = 0.$$

If x is odd, y is even (or vice versa), then the sum of one odd and two even integers can't be 0. If both x, y are even, then divide by 4 as follows:

$$7566271829 \cdot \left(\frac{x}{2}\right)^2 + 19384736483 \cdot \left(\frac{x}{2}\right) \cdot \left(\frac{y}{2}\right) + 362748575 \cdot \left(\frac{y}{2}\right)^2 = 0.$$

Conclude that $\frac{x}{2}, \frac{y}{2}$ form a smaller integer solution and they also should be even. We can continue dividing by 2 without stopping. This infinite division leads to a contradiction that x, y are integers unless $x = y = 0$.

• If given variables are integer, first consider their remainders modulo 2.

Step-by-step hints on Problem 1.16 (other methods are possible)

Step [1]. Is the number 9999!! divisible by 25? How can you justify?

Step [2]. After Step [1] what remainders modulo 100 can 9999!! have?

Step [3]. To find 9999!! modulo 4, consider all odd factors modulo 4.

Step [4]. Any product of 2 consecutive odd factors is $(-1) \pmod 4$.

Step [5]. How many pairs of consecutive odd factors are in 9999!! ?

Step [6]. Use the fact that if integers a, b have the same remainder r modulo $n \geq 2$ (so $a - r$, $b - r$ are divisible by n), then ab is equivalent to r^2 modulo n, namely $ab - r^2 = (a - r)b + r(b - r)$ is divisible by n.